# Black Lenses in Kaluza-Klein Matter

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# Rigidity

Consider a stationary black hole spacetime of dimension $D$ satisfying the vacuum Einstein equations

$$\mathrm{Ric}(g) = \Lambda g.$$

Stationarity gives an asymptotically timelike Killing field with complete orbits, and under some more technical assumptions the rigidity theorem yields additional (rotational) symmetries.

**Theorem (Hollands, Ishibashi, Wald 07; Moncrief, Isenberg 08)**

*An asymptotically flat, analytic stationary vacuum black hole solution to the vacuum Einstein equations admits* $1 \leq N \leq [\frac{D-1}{2}]$ *mutually commuting Killing fields with closed orbits of period* $2\pi$ *such that* $V = \partial_t + \sum_{i=1}^{N} \Omega_i \partial_{\phi^i}$ *is normal to the event horizon for some constants* $\Omega_i$ *referred to as angular velocities.*

In particular the event horizon is a Killing horizon.

# Topology of the Horizon

**Theorem (Galloway, Schoen 06; Galloway 08)**

*Cross-sections of the event horizon in a stationary vacuum spacetime with $\Lambda \geq 0$ are of positive Yamabe invariant, ie. they admit metrics of positive scalar curvature. This result holds more generally for stable MOTS in spacetimes satisfying the dominant energy condition.*

This has strong consequences in $D \leq 5$ but becomes considerably more flexible as dimensions increase. In particular the following topological classification is implied by by the prime decomposition theorem along with the resolution of the Poincaré conjecture.

**Corollary**

*A compact orientable 3-dimensional horizon cross-section in the presence of $\Lambda \geq 0$ is diffeomorphic to a spherical space, $S^1 \times S^2$, or a finite connected sum thereof.*

- This has been refined by Hollands, Holland, and Ishibashi '11 using the rigidity theorem and topological censorship, to show that the connected sums can only occur between copies of $S^1 \times S^2$ and lens spaces $L(p,q)$. Moreover, Hollands and Yazadjiev '11 showed that if the maximum amount of rotational symmetry is assumed (that is $D-3$ rotational Killing fields) then the only possibilities are $S^3 \times T^{D-5}$, $S^2 \times T^{D-4}$, and $L(p,q) \times T^{D-5}$.
- In $D=5$ and $\Lambda=0$, among the possible topologies the sphere $S^3$ has been realized by the Myers-Perry solution (generalization of Kerr '86), the ring $S^1 \times S^2$ by Emparan-Reall (singly spinning '02) and Pomeransky-Senkov (doubly spinning '06). So far vacuum lenses are not known to be geometrically regular, although some have been constructed in minimal supergravity by Kunduri-Lucietti ($\mathbb{RP}^3$ '14) and Tomizawa-Nozawa ($L(p,1)$ '16).
- In the case of extreme black holes, it turns out that more can be said beyond the condition of positive Yamabe invariant.

# Domain of Outer Communication (DOC)

- **Question 1.** What are the possible topologies of the domain of outer communication? The main topology result for the DOC is Topological Censorship, which states that in an AF or AKK spacetime satisfying the null energy condition, any curve with endpoints in the asymptotic region is homotopic to a curve in that region. This together with Hawking's horizon topology theorem imply via the Poincare' conjecture that in 4D the DOC of an AF globally hyperbolic spacetime must be $\mathbb{R} \times \left(\mathbb{R}^3 \setminus \cup_i B_i^3\right)$.
- **Question 2.** Is the DOC topology uniquely determined by the horizon topology and the topology of its asymptotic end?
- **Question 3.** For each possible topology of the DOC does there exist a stationary vacuum solution associated to it? In other words, does each topology support a stationary vacuum solution?

# The Orbit Space in More Detail

**Theorem (Hollands, Yazadjiev 11)**

*The orbit space $M^5/[\mathbb{R}\times U(1)^2]$ is homeomorphic to $\{(\rho,z)\mid\rho>0\}$. The boundary of the orbit space $\rho=0$ consists of a sequence of intervals $(-\infty,z_1)$, $(z_1,z_2)$, ..., $(z_n,\infty)$ on which either $|\partial_t+\Omega_i\partial_{\phi^i}|$ vanishes (horizon rod) or a linear combination $v^i\partial_{\phi^i}$ vanishes (axis rod) where the rod structure $(v^1,v^2)\in\mathbb{Z}^2$.*

Observe that $v_i\in\mathbb{Z}$ since elements of the isotropy subgroup at the axis are of the form $(e^{iv^1\phi},e^{iv^2\phi})$, $0\leq\phi<2\pi$, and the isotropy subgroup forms a proper closed subgroup of $U(1)^2$. That is, the isotropy subgroup yields a simple closed curve in the torus exactly when the slope of its winding is rational.

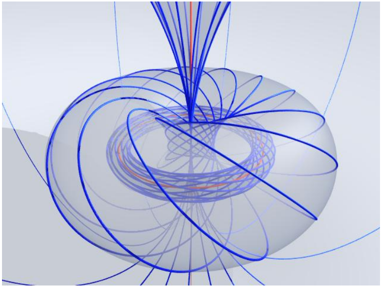
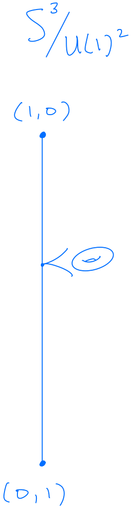
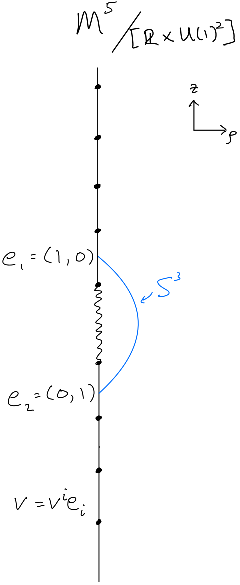


Figure: Hopf Fibration. The round metric on $S^3$ is given by

$$d\theta^2 + \sin^2\theta(d\phi^1)^2 + \cos^2\theta(d\phi^2)^2,$$

$\theta \in [0, \pi/2]$. Most general horizon topologies in this setting are $L(p,q) \times T^{n-2}$ or $S^1 \times S^2 \times T^{n-2}$.



# DOC Topology

**Theorem (MK, Matsumoto, Yamada, Weinstein)**

*The topology of the domain of outer communication of an orientable stationary bi-axisymmetric spacetime satisfying the null energy condition is* $\mathcal{M}^5 = \mathbb{R} \times M^4$ *with the Cauchy surface given by a union of the form*

$$M^4 = \cup_{j=1}^{J} \mathcal{P}\left(\xi_{1,j}, \cdots, \xi_{I_j,j}\right) \cup_{n=1}^{N_1} C_n^4 \cup_{m=1}^{N_2} B_m^4 \cup M_{end}^4,$$

*in which each constituent is a closed manifold with boundary and all are mutually disjoint expect possibly at the boundaries. Here* $\mathcal{P}$ *is a plumbing of disc bundles* $\xi_{i,j}$ *over* $S^2$, $B_m^4$ *is a 4-ball,* $C_n^4$ *is* $B^2 \times S^1 \times [0,1]$, *and* $M_{end}^4$ *is either* $\mathbb{R}_+ \times S^3$, $\mathbb{R}_+ \times S^1 \times S^2$, *or* $\mathbb{R}_+ \times L(p,q)$ *depending on whether the spacetime is asymptotically flat, asymptotically Kaluza-Klein, or asymptotically locally Euclidean.*

# Compactified DOC Topology with Filled in Horizons

**Theorem**

*Consider the domain of outer communication $\mathcal{M}^5 = \mathbb{R} \times M^4$ of an orientable stationary bi-axisymmetric spacetime satisfying the null energy condition, with $H$ horizon cross-sectional components. There exists a choice of horizon fill-ins $\{\tilde{M}_h^4\}_{h=1}^H$ and a cap for the asymptotic end $\tilde{M}_{end}^4$, each of which is either a 4-ball $B^4$ or a plumbed finite sequence of disc bundles over the 2-sphere $\mathcal{P}(\xi_{h_1}, \cdots, \xi_{h_l})$, such that the compactified Cauchy surface*

$$\tilde{M}^4 = \left(M^4 \setminus M_{end}^4\right) \cup_{h=1}^H \tilde{M}_h^4 \cup \tilde{M}_{end}^4$$

*is homeomorphic to the sphere $S^4$, a connected sum of 2-sphere products $\# m S^2 \times S^2$, or a connected sum of complex projective planes $(\# n\mathbb{CP}^2) \# \left(\# \ell \overline{\mathbb{CP}}^2\right)$. Moreover, the disc bundles for each fill-in and cap may be computed algorithmically.*

# Existence Results for Stationary Black Holes

We restrict attention now to vacuum spacetimes $(M^{n+3}, \mathbf{g})$ which are stationary and multi-axisymmetric, that is with isometry group $\mathbb{R} \times U(1)^n$. Frobenius' theorem, the vacuum Einstein equations, and some technical hypotheses imply Weyl-Papapetrou form

$$\mathbf{g} = f^{-1}e^{2\sigma}(d\rho^2 + dz^2) - f^{-1}\rho^2 dt^2 + f_{ij}(d\phi^i + w^i dt)(d\phi^j + w^j dt)$$

where $f = \det f_{ij}$, $\rho^2$ is minus the determinant of the Killing part of the metric, and $\partial_t$, $\partial_{\phi^i}$ are the generators for the symmetry. The interior of the orbit space is the right-half plane

$$M^{n+3}/[\mathbb{R} \times U(1)^n] \cong \mathbb{R}^2_+ = \{(\rho, z) \mid \rho > 0\}.$$

# Einstein-Hilbert Action

- The 1-forms $\chi^i = w^i dt$ on the 3-dimensional Lorentzian orbit space $\hat{M}^3 = M^{n+3}/U(1)^n$ measure the obstruction to the rotational Killing fields being hypersurface orthogonal

$$ff_{ij} *_3 d\chi^j = *(\eta_1 \wedge \cdots \wedge \eta_n \wedge d\eta_i) = d\omega_i.$$

Twist potential functions encode horizon angular momenta

$$\mathcal{J}_i = \frac{1}{8\pi} \int_{\mathcal{H}} *d\eta_i = \frac{\pi}{4}(\omega_i^+ - \omega_i^-).$$

- Following Maison, the dimensionally reduced action is then given by

$$\mathcal{S}[q, \Psi] = \int_{M^3} \left( R_q - 2q^{\mu\nu} G_{AB} \partial_\mu \Psi^A \partial_\nu \Psi^B \right) \text{Vol}_q,$$

where $R_q$ is the scalar curvature of the 3D orbit space metric $f^{-1}q$, $\Psi = (f_{ij}, \omega_i)$, and $G_{AB}$ is a complete nonpositively curved metric on symmetric space $SL(n+1, \mathbb{R})/SO(n+1)$.

The reduced field equations are conveniently expressed as

$$\begin{aligned} \mathrm{Ric}(\gamma)_{\mu\nu} &= \frac{1}{8}\mathrm{Tr}(\Phi^{-1}\partial_\mu\Phi\Phi^{-1}\partial_\nu\Phi), \\ \nabla^\mu(\Phi^{-1}\partial_\mu\Phi) &= 0, \end{aligned}$$

where $\Phi$ is a $(n+1)\times(n+1)$, symmetric, positive definite matrix of determinant 1 constructed from $\Psi$ by

$$\Phi = \begin{pmatrix} f^{-1} & -f^{-1}\omega_i \\ -f^{-1}\omega_i & f_{ij}+f^{-1}\omega_i\omega_j \end{pmatrix}$$

From the matrix polar decomposition $A\in SL(n+1,\mathbb{R})$ may be written $A=PO$ where $P$ is symmetric positive definite and $O\in SO(n+1)$. Then $\Phi$ may be identified with an element $[A]\in SL(n+1,\mathbb{R})/SO(n+1)$ by $\Phi=AA^t$.

# Precise Converse Problem

**Theorem (Hollands, Yazadjiev 08)**

*Asymptotically Kaluza-Kelin stationary vacuum multi-axisymmetric solutions of the Einstein equations are uniquely determined by their rod data: rod structures and potential constants (angular momenta).*

**Theorem (Kakkat, MK, Rainone, Weinstein, Yamada 21)**

*Given potential constants and a rod structure, there exists a corresponding unique singular harmonic map* $\Phi : \mathbb{R}^3 \setminus \Gamma \to SL(n+1, \mathbb{R})/SO(n+1)$ *encoding this data. From this map, an AKK a multi-axisymmetric stationary vacuum black hole solution may be constructed which realizes this data. We can also treat ALE asymptotics and orbifold singularities.*

# Proof

- Construct an approximate solution referred to as the model map $\Phi_0 : \mathbb{R}^3 \setminus \Gamma \to SL(n+1,\mathbb{R})/SO(n+1)$. It encodes the rod data and potential constants, and has finite tension $|\tau(\Phi_0)| < C$ where

$$\tau(\Phi)^l = \Delta\Phi^l + \delta^{ab}\partial_a\Phi^j\partial_b\Phi^k\Gamma^l_{jk}.$$

- Consider an exhaustion of $\mathbb{R}^3 \setminus \Gamma$ by precompact domains $\{\Omega_i\}$, and solve the Dirichlet problem

$$\tau(\Phi_i) = 0 \;\text{ in }\; \Omega_i, \qquad \Phi_i = \Phi_0 \;\text{ on }\; \partial\Omega_i.$$

- Establish $L^\infty$ estimates: maximum principle.
- Local energy bounds: integration by parts.
- Bootstrap: Bochner formula with De Giorgi-Nash-Moser.

# Topology of the Domain of Outer Communication

**Theorem**

*In low dimensions the compactified Cauchy surfaces of the domain of outer communication must have one of the following topologies. Here $k$ indicates the 2nd Betti number and $0 \leq \ell \leq k$.*

$$
\begin{aligned}
D=5: &\quad S^4, \qquad \#\tfrac{k}{2}(S^2\times S^2), \qquad \ell\mathbb{CP}^2\#(k-\ell)\overline{\mathbb{CP}}^2,\\
D=6: &\quad S^5, \qquad \#k(S^2\times S^3), \qquad (S^2\tilde{\times}S^3)\#(k-1)(S^2\times S^3),\\
D=7: &\quad S^3\times S^3, \quad \#k(S^2\times S^4)\#(k+1)(S^3\times S^3),\\
&\quad (S^2\tilde{\times}S^4)\#(k-1)(S^2\times S^4)\#(k+1)(S^3\times S^3).
\end{aligned}
$$

**Conjecture**

*A compactified Cauchy surface of the DOC, in the spin case, is homeomorphic to a connected sum of products of spheres.*

# Remaining Issues

- *Analytic regularity.*
  The metric coefficients in the Weyl-Papapetrou form should be shown to be smooth and up to the axis $\Gamma$. This was achieved in the 4D case independently by Tian-Li '92 and Weinstein '90. Later Nguyen '11 extended this to the Einstein-Maxwell setting.
- *Conical singularities.*
  In addition to the analytic regularity, conical singularities on axis rods should be ruled out for various ranges of the parameters. A conical singularity at a point on an axis rod is measured by the angle deficiency $\theta \in (-\infty, 2\pi)$ given by

$$\frac{2\pi}{2\pi - \theta} = \lim_{\rho \to 0} \frac{2\pi \cdot \mathrm{Radius}}{\mathrm{Circumference}} = \lim_{\rho \to 0} \sqrt{\frac{\rho^2 f^{-1} e^{2\sigma}}{f_{ij} v^i v^j}},$$

where $(v^1, v^2)$ is the associated rod structure. Absence of a conical singularity is characterized by a zero angle deficiency.

**Theorem (MK, Rainone 23)**

*Any possible topology of the domain of outer communication for a multi-axisymmetric spacetime of dimension greater than or equal to 4, is realizable by a regular static solution of the Einstein equations with Kaluza-Klein matter. In particular, these solutions are obtained from a higher dimensional asymptotically Kaluza-Klein vacuum solution by dimensional reduction on tori.*

- Given the desired DOC $\mathcal{M}^{n+3}$ for a $(n+3)$-dimensional static/stationary spacetime admitting $U(1)^n$ symmetry with $n \geq 1$, we show how to encode its topology in a higher dimensional DOC $\tilde{\mathcal{M}}^{n+3+k}$ having a relatively simple topological structure.
- On this higher dimensional spacetime manifold, we solve the static/stationary vacuum Einstein equations with $U(1)^{n+k}$ symmetry, and take advantage of the simple topology to balance any conical singularities.
- A dimensional reduction, or quotient procedure, is then carried out in order to obtain a regular solution with Kaluza-Klein matter on the original topology $\mathcal{M}^{n+3}$.

# Encode Topology in Higher Dimensions

- Consider a topology $\mathcal{M}^{n+3}$ that we wish to realize as the DOC for a static solution of the Einstein equations. It may be characterized by a $n$-dimensional rod data set $\mathcal{D} = \{(\mathbf{v}_1, \Gamma_1), \ldots, (\mathbf{v}_k, \Gamma_k)\}$.
- Define a new $(n+k)$-dimensional rod data set $\tilde{\mathcal{D}} = \{(\tilde{\mathbf{v}}_1, \Gamma_1), \ldots, (\tilde{\mathbf{v}}_k, \Gamma_k)\}$ by $\tilde{\mathbf{v}}_i = \bar{\mathbf{v}}_i + \mathbf{e}_{n+i}$ for $i = 1, \ldots, k$, where $\mathbf{e}_j$ is an element of the standard basis for $\mathbb{Z}^{n+k}$ having 1 in position $j$ and zeros elsewhere, and $\bar{\mathbf{v}}_i = (\mathbf{v}_i, \mathbf{0}) \in \mathbb{Z}^{n+k}$.
- Note that since $\det_2(\tilde{\mathbf{v}}_i, \tilde{\mathbf{v}}_j)$ divides $\det_2(\mathbf{v}_i, \mathbf{v}_j)$ the data set $\tilde{\mathcal{D}}$ inherits the admissibility property from $\mathcal{D}$. In particular, the new data yields a manifold $\tilde{\mathcal{M}}^{n+k+3}$ admitting an effective action by $U(1)^{n+k}$.

# Simplicity of Higher Dimensional Topology

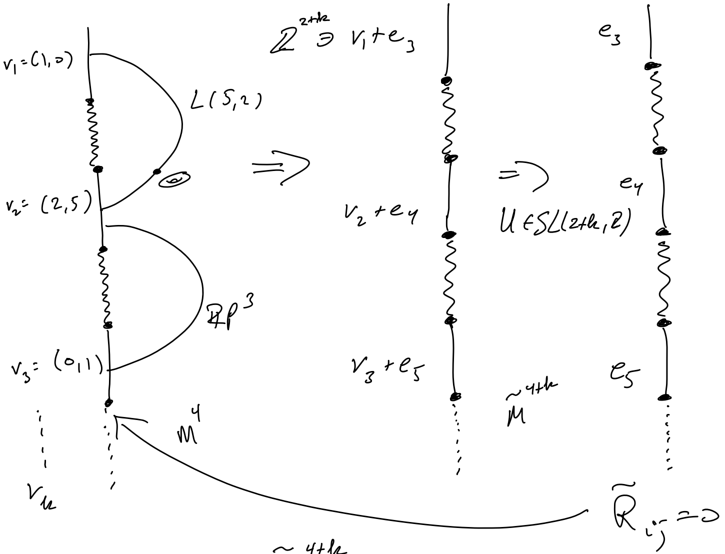
- Rod structures of $\tilde{\mathcal{M}}^{n+k+3}$ consist only of $\mathbf{e}_i$ (in nice coords).
- To see this, we exhibit a coordinate change on the torus fibers $T^{n+k} = \mathbb{R}^{n+k}/\mathbb{Z}^{n+k}$ ie. a matrix $U \in SL(n+k, \mathbb{Z})$ such that defined by $U(\mathbf{e}_j) = \mathbf{e}_j$ for $j = 1, \ldots, n$ and $U(\tilde{\mathbf{v}}_i) = \mathbf{e}_{n+i}$ for $i = 1, \ldots, k$ so that

$$U^{-1} = \begin{bmatrix} I_n & V \\ 0 & I_k \end{bmatrix},$$

  where $I_n$ is the $n \times n$ identity matrix, and $V$ is the $n \times k$ matrix consisting of the rod structures $[\mathbf{v}_1, \ldots, \mathbf{v}_k]$.
- Thus, after this coordinate change the rod data set becomes $\tilde{\mathcal{D}}' = \{(\mathbf{e}_{n+1}, \Gamma_1), \ldots, (\mathbf{e}_{n+k}, \Gamma_k)\}$, which implies that the topology of the compactified manifold $\tilde{M}_c^{n+k+2}$ is given by

$$\left[ \mathop{\#}_{\ell=1}^{k-3} \ell \binom{k-2}{\ell+1} S^{2+\ell} \times S^{k-\ell} \right] \times T^n$$

  for $k \geq 4$, whereas for $k = 2, 3$ the topology is $S^{k+2} \times T^n$.




$$R_{ij} - \frac{1}{2} R g_{ij} = 8\pi T_{ij}$$

↑ Kaluza-Klein matter

$$\tilde{M}^{4+k} / T^k$$

$$\partial \tilde{C} \cong \left[ \#_{l=1}^{k-3} \, l \binom{k-2}{l+1} S^{2+l} \times S^{k-l} \right] \times T^2$$

# Solving the Vacuum Equations on $\tilde{\mathcal{M}}^{n+k+3}$

- Consider the ansatz

$$\tilde{\mathbf{g}} = -\rho^2 e^{-\sum_{i=1}^{n+k} u_i} dt^2 + e^{2\alpha}(d\rho^2 + dz^2) + \sum_{i=1}^{n+k} e^{u_i} \left(d\psi^i\right)^2,$$

  where all coefficients depend only on $\rho > 0$, $z$ and the Killing fields $\partial_{\psi^i}$ generate the $U(1)^{n+k}$ rotational isometries with $0 \leq \psi^i < 2\pi$.
- The static vacuum Einstein equations reduce to finding $n+k$ axisymmetric harmonic functions $u_i$ on $\mathbb{R}^3 \setminus \Gamma$. Observe that the ansatz implies axes can only exhibit rod structures of type $\mathbf{e}_i$, $i = 1, \ldots, n+k$.
- For an axis rod $\Gamma_l$ having the rod structure $\mathbf{e}_l$, we find that the corresponding logarithmic angle defect is given by

$$\mathbf{b}_l = \lim_{\rho \to 0} \left( \log \rho + \alpha - \frac{1}{2} u_l \right).$$

# Balancing Conical Singularities

The spacetime $(\tilde{\mathcal{M}}^{n+k+3}, \tilde{\mathbf{g}})$ has the desired topology, satisfies the static vacuum equations, and is asymptotically Kaluza-Klein. However, it may possess conical singularities along axis rods. Nevertheless, due to the diagonal matrix structure of the torus fiber metrics, any conical singularity along an axis rod $\Gamma_i$ may be resolved by adding an appropriate constant to the associated harmonic function $u_{n+i} \mapsto u_{n+i} + c_i$, where the constant $c_i$ is chosen to ensure that the logarithmic angle defect $\mathbf{b}_i = 0$. This translation in the harmonic functions does not alter any of the properties listed above for the spacetime. We note that a related balancing procedure was employed by Emparan-Reall in 2002 for certain examples. Furthermore, absence of conical singularities leads to full regularity of the spacetime metric.

## Theorem

*For each pair of integers $n \geq 0$ and $k \geq 2$, the compactified domain of outer communication topology $\tilde{M}_c^{n+k+2}$ is realized by time slices of a regular, asymptotically Kaluza-Klein (or asymptotically flat when $n = 0$, $k = 2$), static vacuum solution with up to $k-1$ horizons of topology $S^3 \times T^{n+k-2}$.*

When there are no horizons, $\sum_{i=1}^{n+k} u_i - 2\log\rho - c$ is harmonic on $\mathbb{R}^3 \setminus \Gamma$ for some constant $c$, it tends to zero at infinity, and remains bounded upon approach to $\Gamma$. The maximum principle (Weinstein Lemma) implies it vanishes. Hence the static potential is constant.

## Corollary

*For each pair of integers $n \geq 0$ and $k \geq 2$, the topology $\tilde{M}_c^{n+k+2}$ gives rise to a gravitational instanton. More precisely, on the complement of a $B^4 \times T^{n+k-2}$ this manifold admits a regular, complete, Ricci flat Riemannian metric which is asymptotically Kaluza-Klein (or asymptotically flat when $n = 0$, $k = 2$).*

# Dimensional Reduction

- Perform dimensional reduction (quotient procedure) on the constructed static spacetime $\tilde{\mathcal{M}}^{n+k+3}$ having rod data set $\tilde{\mathcal{D}}$.
- Reduce on the free subtorus action which rotates the last $k$ circles in the fibers of $\tilde{\mathcal{M}}^{n+k+3}$ which are parameterized by $(\phi^{n+1}, \ldots, \phi^{n+k})$. The projection map $\tilde{\mathcal{M}}^{n+k+3} \to \tilde{\mathcal{M}}^{n+k+3}/T^k$ may be described by

$$(\mathbf{p}, \phi^1, \ldots, \phi^{n+k}) \mapsto (\mathbf{p}, \phi^1, \ldots, \phi^n),$$

  where $\mathbf{p} \in \mathbb{R}^2_+$.
- The quotient space is homeomorphic to the given topology $\mathcal{M}^{n+3}$, since the projection map implies that the rod data set $\tilde{\mathcal{D}}$ encoding the higher dimensional topology, descends down to the rod data set $\mathcal{D}$ for $\tilde{\mathcal{M}}^{n+k+3}/T^k$.
- Since the action is by isometries, and the static vacuum total space $\tilde{\mathcal{M}}^{n+k+3}$ is regular, the same is true of the quotient $\mathcal{M}^{n+3}$. In particular, there are no conical singularities.

The metric on $\tilde{\mathcal{M}}^{n+k+3}$ may be expressed in Kaluza-Klein format

$$\tilde{\mathbf{g}} = \mathbf{g} + \sum_{\mu,\nu=n+1}^{n+k} h_{\mu\nu}(d\phi^{\mu} + A^{\mu}_{i} d\phi^{i})(d\phi^{\nu} + A^{\nu}_{j} d\phi^{j}),$$

where the desired quotient metric is

$$\mathbf{g} = -(fh)^{-1}\rho^2 dt^2 + (fh)^{-1} e^{2\sigma}(d\rho^2 + dz^2) + \sum_{i,j=1}^{n} f_{ij} d\phi^i d\phi^j,$$

with $f_{ij} + h_{\mu\nu}A^{\mu}_{i}A^{\nu}_{j} = \tilde{f}_{ij}$, $f = \det(f_{ij})$, and $h = \det(h_{\mu\nu})$. The dimensionally reduced Lagrangian on $\mathcal{M}^{n+3}$ may then be expressed

$$\mathcal{L} = \sqrt{hg}\left(R - \frac{1}{4}(|\mathrm{Tr}(h^{-1}\nabla h)|^2 + \mathrm{Tr}(h^{-1}\nabla h)^2 + |\mathcal{F}|^2)\right),$$

where $R$ is the scalar curvature of $\mathbf{g}$, $|\mathcal{F}|^2 = h_{\mu\nu}\mathcal{F}^{\mu\mathbf{ij}}\mathcal{F}^{\nu}_{\mathbf{ij}}$ with $\mathcal{F}^{\mu} = dA^{\mu}$ and $\mathbf{i}, \mathbf{j}$ labelling the coordinates of $\mathbf{g}$, and $g = -\det \mathbf{g}$.

## Theorem (MK, Reiris, Weinstein, Yamada 22)

*There exist regular bi-axisymmetric solutions of the 5D static vacuum Einstein equations, balancing an infinite number of spherical $S^3$ and ring $S^1 \times S^2$ black holes. These solutions are space-periodic, asymptotically Kasner, and have DOC topology $M^5 = \mathbb{R} \times M^4$ where ($i_0, j_0 \in \{0, \infty\}$, 0 indicating empty sum)*

$$M^4 = \#^\infty S^2 \times S^2 \setminus \left[ \cup_{i=1}^{i_0} B_i^4 \cup_{j=1}^{j_0} (S^2 \times B^2)_j \right].$$

## Theorem

- *There exist regular complete bi-axisymmetric solutions of the 5D static vacuum Einstein equations, which are devoid of black holes. These solutions are space-periodic, asymptotically Kasner, and have DOC topology $M^5 = \mathbb{R} \times M^4$ where $M^4 = \#^\infty S^2 \times S^2$.*
- *The time slice is a complete Ricci flat Riemannian manifold of infinte topological type and generic holonomy.*

# Infinite Connected Sum of Sphere Products

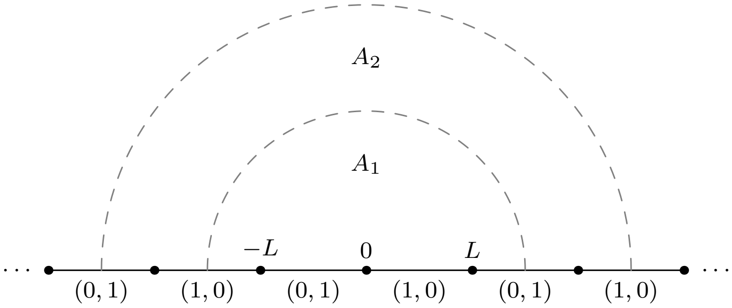



Figure: $\#^{\infty} S^2 \times S^2$

# Higher Dimensions

**Theorem**

- *For each* $n \geq 2$*, there is a regular* $n$*-axisymmetric static vacuum soliton spacetime* $(M^{n+3}, \mathbf{g})$ *which is space-periodic and asymptotically Kasner. The rod structure is periodic with fundamental period* $\mathbf{e}_1, \ldots, \mathbf{e}_n$*. These spacetimes are simply connected and of infinite topological type:* $b_n(M^{n+3}) = \infty$*.*
- *Taking space quotients gives further solutions. For* $n \geq 4$*, the spatial slice is homeomorphic to*

$$\left[\mathop{\#}_{k=1}^{n-3} k\binom{n-2}{k+1} S^{2+k} \times S^{n-k}\right] \setminus (B^2 \times T^n).$$

- *Time slices are complete Ricci flat Riemannian manifolds. They may be Wick rotated to produce static vacuum black hole solutions, having an infinite number* $S^3 \times T^{n-2}$ *horizons.*

Kasner asymptotics refers to the fact that asymptotically, after making the coordinate change $\tau = \rho^{C+1}$, the spacetime metric satisfies

$$\mathbf{g} \sim -4L^2 dt^2 + \frac{1}{(C+1)^2} d\tau^2 + \tau^{\frac{2C}{C+1}} dz^2 + \sum_{i=1}^{n} \tau^{\frac{A_i}{C+1}} \left(d\phi^i\right)^2 .$$

The coefficients of the powers of $\tau$ satisfy the Kasner conditions since the metric is Ricci flat. Recall that the Kasner metric on $\mathbb{R}^{n+1,1}$ takes the form

$$g_K = -dt^2 + \sum_{i=0}^{n} t^{2p_i} \left(dx^i\right)^2 ,$$

and that this metric satisfies the vacuum Einstein equations exactly when the Kasner conditions hold:

$$\sum_{i=0}^{n} p_i = 1, \qquad \sum_{i=0}^{n} p_i^2 = 1.$$